Matrix :

A matrix is an ordered rectangular array of numbers or functions.

The numbers or functions are called the elements or the entries of the matrix. It is denoted by capital letter. It is denoted by ( ), [ ], || ||.

$$A = \begin{bmatrix} 8 & 10 \\ 10 & 15 \\ 18 & 20 \end{bmatrix}$$

Order of a matrix

A matrix having m rows and n columns is called a matrix of order m×n (read as an m by n matrix).

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \cdots & a_{1j} \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{2j} \cdots & a_{2n} \\ \vdots & \vdots & & & \\ a_{i1} & a_{i2} & a_{i3} \cdots & a_{ij} \cdots & a_{in} \\ \vdots & \vdots & & & \\ a_{m1} & a_{m2} & a_{m3} \cdots & a_{mj} \cdots & a_{mn} \end{bmatrix}$$

Q. Construct a 3×2 matrix whose elements are given by $a_{ij} = \frac{1}{2}|i - 3j|$.

⇒

In general a 3×2 matrix is given by $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$

Now $a_{ij} = \frac{1}{2}|i - 3j|$, $i = 1, 2, 3$ and $j = 1, 2$.

Therefore. $a_{11} = \frac{1}{2}|1 - 3 \times 1| = 1.$

$a_{12} = \frac{1}{2}|1 - 3 \times 2| = \frac{5}{2}$

$a_{21} = \frac{1}{2}|2 - 3 \times 1| = \frac{1}{2}$

$a_{22} = \frac{1}{2}|2 - 3 \times 2| = \frac{4}{2} = 2.$

$a_{31} = \frac{1}{2}|3 - 3 \times 1| = 0.$

$a_{32} = \frac{1}{2}|3 - 3 \times 2| = \frac{3}{2}.$

Hence the required matrix is given by

$$A = \begin{bmatrix} 1 & 5/2 \\ 1/2 & 2 \\ 0 & 3/2 \end{bmatrix}$$

Types of Matrices:

(i) Column matrix

A matrix is said to be a column matrix if it has only one column.

$$\begin{array}{cc} A = \begin{bmatrix} 8 \\ 15 \\ 20 \end{bmatrix}_{3 \times 1} & \begin{array}{c} \rightarrow R_1 \\ \rightarrow R_2 \\ \rightarrow R_3 \end{array} \\ \downarrow & \\ C_1 & \end{array}$$

(ii) Row matrix :

A matrix is said to be a row matrix if it has only one row.

$$A = \begin{bmatrix} 1 & 4 & 6 \end{bmatrix}_{1\times 3} \rightarrow R_1$$

$$\begin{matrix} \downarrow & \downarrow & \downarrow \\ C_1 & C_2 & C_3 \end{matrix}$$

(iii) Square matrix:

A matrix in which the number of rows are equal to the number of columns, is said to be a square matrix.

$$A = \begin{bmatrix} 8 & 13 & 5 \\ 3 & 2 & 15 \\ 18 & 20 & 13 \end{bmatrix}_{3\times 3}$$

$\boxed{\text{where } m = n}$

(iv) Diagonal matrix.

A square matrix $B = [b_{ij}]_{m\times n}$ is said to be a diagonal matrix if all its non-diagonal elements are zero, that is a matrix $B = [b_{ij}]_{m\times n}$ is said to be a diagonal matrix if $b_{ij} = 0$, when $i \neq j$.

(v) Scalar matrix

A diagonal matrix is said to be a scalar matrix if its diagonal elements are equal.

$$A = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

(vi) Identity matrix:

A square matrix in which elements in the diagonal are all 1 and rest are all zero is called an identity matrix.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(vii) Zero matrix:

A matrix is said to be zero matrix or null matrix if all its elements are zero.

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Equality of matrices.

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be equal if.

(i) they are of the same order.

(ii) each element of A is equal to the corresponding element of B, ~~tha~~ that is $a_{ij} = b_{ij}$ for all i and j.

$\therefore \boxed{A = B}$ $\quad A = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$

Operations on Matrices.

Addition of matrices.

Q. $A = \begin{bmatrix} \cos^2 x & \sin^2 x \\ \sin^2 x & \cos^2 x \end{bmatrix}$, $B = \begin{bmatrix} \sin^2 x & \cos^2 x \\ \cos^2 x & \sin^2 x \end{bmatrix}$

$$A+B = \begin{bmatrix} \cos^2 x + \sin^2 x & \sin^2 x + \cos^2 x \\ \sin^2 x + \cos^2 x & \cos^2 x + \sin^2 x \end{bmatrix}$$

$$A+B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$ Ans

Properties of matrix addition.

(i) Commutative Law:

If $A = [a_{ij}]$, $B = [b_{ij}]$ are matrices of the same order, say $m \times n$, then $A+B = B+A$.

eg. → $[a_{ij}] + [b_{ij}] = [b_{ij}] + [a_{ij}]$.

(ii) Associative law:

For any three matrices $A = [a_{ij}]$, $B = [b_{ij}]$, $C = [c_{ij}]$ of the same order, say $m \times n$, then

$$(A+B)+C = A+(B+C)$$

$$([a_{ij}] + [b_{ij}]) + [c_{ij}] = [a_{ij}] + ([b_{ij}] + [a_{ij}])$$

(iii) Existence of additive identity.

~~Let $A = [a_{ij}]$ be an $m \times n$ matrix, then we have another matrix as $-A = [-a_{ij}]$ $m \times n$ such that $A +$~~

Let $A = [a_{ij}]$ be an $m \times n$ matrix and O be an $m \times n$ zero matrix, then $A + O = O + A = A$. In other words, O is the additive identity for matrix addition.

(iv) The Existence of additive inverse:

Let $A = [a_{ij}]_{m \times n}$ be any matrix, then we have another matrix as $-A = [-a_{ij}]_{m \times n}$ such that $A + (-A) = (-A) + A = O$. So $-A$ is the additive inverse of A or negative of A.

$$8 + (-8) = 0 = (-8) + 8.$$

Multiplication of Matrices:

$$A \times B = \begin{bmatrix} 3 & 8 \\ 2 & 10 \end{bmatrix} \times \begin{bmatrix} 5 & 7 \\ 4 & 15 \end{bmatrix}$$

$$= \begin{bmatrix} 3\times5 + 8\times4 & 3\times7 + 8\times15 \\ 2\times5 + 10\times4 & 2\times7 + 10\times15 \end{bmatrix}$$

$$= \begin{bmatrix} 15+32 & 21+120 \\ 10+40 & 14+150 \end{bmatrix}$$

$$= \begin{bmatrix} 47 & 141 \\ 50 & 164 \end{bmatrix}$$

Definition.

The product of two matrices, A and B defined if the number of columns of A is equal to the no. of rows of B. Let $A = [a_{ij}]$ be an $m \times n$ matrix and $B = [b_{jk}]$ be an $n \times p$ matrix. Then the product of the matrices A and B is the matrix C of order $m \times p$.

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Q. $A = \begin{bmatrix} 3 & 2 \\ 8 & 6 \end{bmatrix}_{2\times 4}$

$B = \begin{bmatrix} 2 & 8 & 3 & 4 \\ 3 & 4 & 8 & 7 \end{bmatrix}_{2\times 4}$.

$A \times B = \begin{bmatrix} 6+6 & 24+8 & 9+16 & 12+14 \\ 16+18 & 64+24 & 24+48 & 32+42 \end{bmatrix}$

$A \times B = \begin{bmatrix} 12 & 32 & 25 & 26 \\ 34 & 88 & 72 & 74 \end{bmatrix}_{2\times 4}$.

Transpose of matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{21} & a_{23} \end{bmatrix}$$

$$A' \text{ or } A^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{bmatrix}$$

$$A = \begin{bmatrix} 8 & 3 & 28 \\ 3 & 18 & 13 \end{bmatrix}$$

$$A' = \begin{bmatrix} 8 & 3 \\ 3 & 18 \\ 28 & 13 \end{bmatrix}$$

Definition.

If $A = [a_{ij}]$ be an $m \times n$ matrix, then the matrix obtained by interchanging the rows and columns of A is called the transpose of A. Transpose of the matrix A is denoted by A' or (A^T).

In other words, if $A = [a_{ij}]_{m \times n}$, then

$$A' = [a_{ji}]_{n \times m}.$$

$$A = \begin{bmatrix} 8 & 3 & 18 \\ 3 & 28 & 13 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 8 & 3 \\ 3 & 28 \\ 18 & 13 \end{bmatrix}$$

Property of Transpose of matrices.

For any matrices A and B is of suitable orders, we have.

(i) $A = (A')'$

If $A = \begin{bmatrix} 3 & 2 \\ 5 & 6 \\ 7 & 8 \end{bmatrix}$

Now $A' = \begin{bmatrix} 3 & 5 & 7 \\ 2 & 6 & 8 \end{bmatrix}$

then $(A')' = \begin{bmatrix} 3 & 2 \\ 5 & 6 \\ 7 & 8 \end{bmatrix}$

(ii) $(KA)' = KA'$ where K is constant.

$$A = \begin{bmatrix} 5 & 6 & 13 \\ 3 & 8 & 10 \end{bmatrix}$$

$$KA = \begin{bmatrix} 5K & 6K & 13K \\ 3K & 8K & 10K \end{bmatrix}$$

$$(KA)' = \begin{bmatrix} 5K & 3K \\ 6K & 8K \\ 13K & 10K \end{bmatrix}$$

$$= k \begin{bmatrix} 5 & 3 \\ 6 & 8 \\ 13 & 10 \end{bmatrix}$$

$$= KA'$$

Then $A' = \begin{bmatrix} 5 & 3 \\ 6 & 8 \\ 13 & 10 \end{bmatrix}$

$$\Rightarrow \boxed{(KA)' = KA'}$$

3. $(A+B)' = A' + B'$

$A = \begin{bmatrix} 13 & 8 \\ 5 & 10 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 5 \\ 6 & 7 \end{bmatrix}$

$A' = \begin{bmatrix} 13 & 5 \\ 8 & 10 \end{bmatrix}$, $B' = \begin{bmatrix} 3 & 6 \\ 5 & 7 \end{bmatrix}$

$$L.H.S = \begin{bmatrix} 13 & 8 \\ 5 & 10 \end{bmatrix} + \begin{bmatrix} 3 & 5 \\ 6 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & 13 \\ 11 & 17 \end{bmatrix}$$

$$R.H.S = A' + B$$

$$= \begin{bmatrix} 13 & 5 \\ 8 & 10 \end{bmatrix} + \begin{bmatrix} 3 & 6 \\ 5 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & 11 \\ 13 & 17 \end{bmatrix}$$

$R.H.S = L.H.S$ proved.

4. $(AB)' = B'A'$

$$A = \begin{bmatrix} 5 & 6 \\ 3 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} 10 & 15 \\ 2 & 4 \end{bmatrix}$$

$$A' = \begin{bmatrix} 5 & 3 \\ 6 & 7 \end{bmatrix}, \quad B' = \begin{bmatrix} 10 & 2 \\ 15 & 4 \end{bmatrix}$$

$$AB = \begin{bmatrix} 5 & 6 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} 10 & 15 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 50+12 & 75+24 \\ 30+14 & 45+28 \end{bmatrix} = \begin{bmatrix} 62 & 99 \\ 44 & 73 \end{bmatrix}$$

$$L.H.S = (AB)' = \begin{bmatrix} 62 & 44 \\ 99 & 73 \end{bmatrix}$$

$$R.H.S = \begin{bmatrix} 10 & 2 \\ 15 & 4 \end{bmatrix} \begin{bmatrix} 5 & 3 \\ 6 & 7 \end{bmatrix} = \begin{bmatrix} 50+12 & 30+14 \\ 75+24 & 45+28 \end{bmatrix} = \begin{bmatrix} 62 & 44 \\ 99 & 73 \end{bmatrix}$$

$L.H.S = R.H.S$ proved.

Theorem 1:

For any square matrix A with real numbers entities, $A + A'$ is a symmetric matrix and $A - A'$ is a skew symmetric matrix.

Let $P = A + A'$

Now $P' = (A + A')'$

$\Rightarrow P' = A' + (A')'$

$\Rightarrow P' = A' + A$ (For commutative law)

$\Rightarrow P' = P$

$\Rightarrow P = P'$

Hence $A + A'$ is symmetric Matrix.

Let $Q = A - A'$

Now $Q' = (A - A')'$

$\Rightarrow Q' = A' - (A')'$

$\Rightarrow Q' = A' - A$

$\Rightarrow Q' = -(-A' + A)$

$\Rightarrow Q' = -(A - A')$ (From commutative law)

$\Rightarrow Q' = -Q$

Therefore $A - A'$ is a skew symmetric matrix.

Theorem 2:

Any square matrix can be expresed as the sum of a symmetric and a skew symmetric matrix.

Let A be a square matrix, then we can write:

$$A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A').$$

Q.10 $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ $A' = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

$$A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$$

$$\Rightarrow A = \frac{1}{2}\begin{bmatrix} 12 & -4 & 4 \\ -2 & 6 & 2 \\ 2 & 2 & 6 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & 1 \\ 2 & 1 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$ Ans

Elementary operation (Transformation) of Matrix.

1. The interchange of any two rows or two columns. Symbollically the interchange of ith and j^{th} rows is denoted by $R_i \leftrightarrow R_j$ and interchange of i^{th} and j^{th} column is denoted by $C_i \leftrightarrow C_j$.

For example :-

(a) $A = \begin{array}{c} \begin{bmatrix} 8 & 3 \\ 11 & 18 \end{bmatrix} \end{array} \begin{array}{l} \rightarrow R_1 \\ \rightarrow R_2 \end{array}$

$$R_1 \leftrightarrow R_2$$

$$A = \begin{bmatrix} 11 & 18 \\ 8 & 3 \end{bmatrix}$$

(b) $A = \begin{array}{cc} \begin{bmatrix} 8 & 3 \\ 11 & 18 \end{bmatrix} \\ \begin{array}{cc} \downarrow & \downarrow \\ C_1 & C_2 \end{array} \end{array}$

$$C_1 \leftrightarrow C_2$$

$$A = \begin{bmatrix} 3 & 8 \\ 18 & 11 \end{bmatrix}$$

2. The multiplication of the elements of any row or column by a non zero number.

Symbolically, the multiplication of each element of the i^{th} row by k, where $k \neq 0$ is denoted by $R_i \to kR_i$.

$$\boxed{C_i \to kC_i}$$

(a)
$$A = \begin{bmatrix} 8 & 3 \\ 11 & 18 \end{bmatrix} \begin{array}{l} \to R_1 \\ \to R_2 \end{array}$$

$$R_2 \to kR_2$$

$$A = \begin{bmatrix} 8 & 3k \\ 11 & 18k \end{bmatrix}$$

where k is non-zero constant.

(b)
$$A = \begin{array}{c} \begin{bmatrix} 8 & 3 \\ 11 & 18 \end{bmatrix} \\ \begin{array}{cc} \downarrow & \downarrow \\ C_1 & C_2 \end{array} \end{array}$$

$$\boxed{C_1 \to kC_1}$$

$$A = \begin{bmatrix} 8k & 3 \\ 11k & 18 \end{bmatrix}$$

where k is non-zero constant.

3. The addition to the elements of any row or column, the corresponding elements of any other row or column multiplied by any non zero number.

Symbolically, the addition to the elements of i^{th} row, the corresponding elements of j^{th} row multiplied by k is denoted by $R_i \to R_i + kR_j$.

Example :-

(a) $A = \begin{bmatrix} 8 & 3 \\ 11 & 18 \end{bmatrix} \begin{matrix} \to R_1 \\ \to R_2 \end{matrix}$

$$R_1 \to R_1 + kR_2$$

$$A = \begin{bmatrix} 8+11k & 3+18k \\ 11 & 18 \end{bmatrix}$$

(b) $A = \begin{bmatrix} 8 & 3 \\ 11 & 18 \end{bmatrix}$
$\quad\quad\quad\downarrow \quad \downarrow$
$\quad\quad\quad C_1 \quad C_2$

$$C_2 \to C_2 + kC_1$$

$$A = \begin{bmatrix} 8 & 3+8k \\ 11 & 18+11k \end{bmatrix}$$

Inverse of matrices.

If A is a square matrix of order m, and if there exists another square matrix B of the same order m, such that $AB = BA = I$, then B is called the inverse matrix of A and it is denoted by A^{-1}.

eg. $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ $\quad B = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

$$AB = \begin{bmatrix} 2\times 2 + 3\times -1 & 2\times 3 + 2\times 3 \\ 1\times 2 + 2\times -1 & 1\times 3 + 2\times 2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$BA = \begin{bmatrix} 4-3 & -6+6 \\ -2+2 & -3+4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Theorem 4.

If A and B are ~~matrices~~ invertible matrices of the same order, then $(AB)^{-1} = B^{-1}A^{-1}$

We know that :-

$$(AB)(AB)^{-1} = I$$

Multiplying A^{-1} on both side, we have.

$$\Rightarrow A^{-1}(AB)(AB)^{-1} = A^{-1} \cdot I$$

$$\Rightarrow (A^{-1}A)B(AB)^{-1} = A^{-1}$$

$$\Rightarrow IB(AB)^{-1} = A^{-1}$$

$$\Rightarrow B(AB)^{-1} = A^{-1}$$

Multiplication of B^{-1} on both side, we have,

$$B^{-1}B(AB)^{-1} = B^{-1}A^{-1}$$

$$I(AB)^{-1} = B^{-1}A^{-1}$$

$$(AB)^{-1} = B^{-1}A^{-1}$$ proved.

Q.1.→ Let $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

We know that ow

$$A = AI$$

$$\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$$

By Row transformation.

$$R_2 \to R_2 - 2R_1$$

$$\begin{bmatrix} 1 & -1 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} A$$

$$R_2 \to \frac{1}{5} \times R_2$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2/5 & 1/5 \end{bmatrix} A$$

$$R_1 \to R_1 + R_2$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3/5 & 1/5 \\ -2/5 & 1/5 \end{bmatrix} A$$

Therefore, $A^{-1} = \begin{bmatrix} 3/5 & 1/5 \\ -2/5 & 1/5 \end{bmatrix}$ Ans.